Detecting chaos in hydrodynamics

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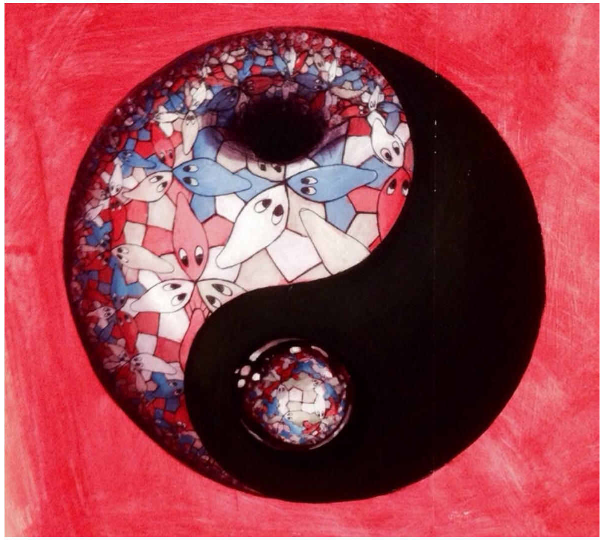

Netherlands Organisation for Scientific Research

Delta Institute for Theoretical Physics

FOM

Saso Grozdanov

Vincenzo Scopelliti

Chaos and hydrodynamics

- Hydrodynamics from the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t} \right)_{\rm coll}$$

Here $f = f(\mathbf{x}, \mathbf{p}, t)$ one-particle distribution function

- Moments of the Boltzmann equation give Navier-Stokes

$$\int d\mathbf{p} m f(\mathbf{x}, \mathbf{p}, t) = \rho(\mathbf{x}, t) \qquad \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

$\mathbf{F} = 0$

$$\int d\mathbf{p} \mathbf{p} f(\mathbf{x}, \mathbf{p}, t) = m\mathbf{v}(\mathbf{x}, t) \qquad \partial_t (\rho \mathbf{v}_i) + \nabla_j (\rho \mathbf{v}_j \mathbf{v}_i + P_{ij}) = 0$$

- The Boltzmann equation from statistical mechanics

  The k-particle distribution function

$$f_k = f(\mathbf{x}_1, \mathbf{p}_1, \mathbf{x}_2, \mathbf{p}_2, \ldots, \mathbf{x}_k, \mathbf{p}_k, t)$$

  Time-evolution governed by BBGKY hierarchy

$$U = H_{int}$$

$$\frac{d}{dt} f_n = \int d^3 q_{n+1} d^3 p_{n+1} \sum_{i=1}^{n} \{U, f_{n+1}\}_{\text{PB wrt } q_i, p_i}$$

- Truncation of the BBGKY hierarchy

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Assumption of molecular chaos

$$f_2 \sim f_1^2$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f = \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \sigma(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_2, \mathbf{p}_3) \left(f(\mathbf{p}_2, t) f(\mathbf{p}_3, t) - f(\mathbf{p}, t) f(\mathbf{p}_1, t) \right)$$

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- Linearized Boltzmann equation

$$\frac{d}{dt} f(\mathbf{p}, t) = \int_{\mathbf{k}} (R^{in}(\mathbf{p}, \mathbf{k}) - R^{out}(\mathbf{p}, \mathbf{k})) f(\mathbf{k}, t)$$

- Transport from the Boltzmann equation

Maxwell

$$\eta = \frac{1}{3} m \rho \ell_{\rm m.f.p.} \sqrt{\langle v^2 \rangle}$$

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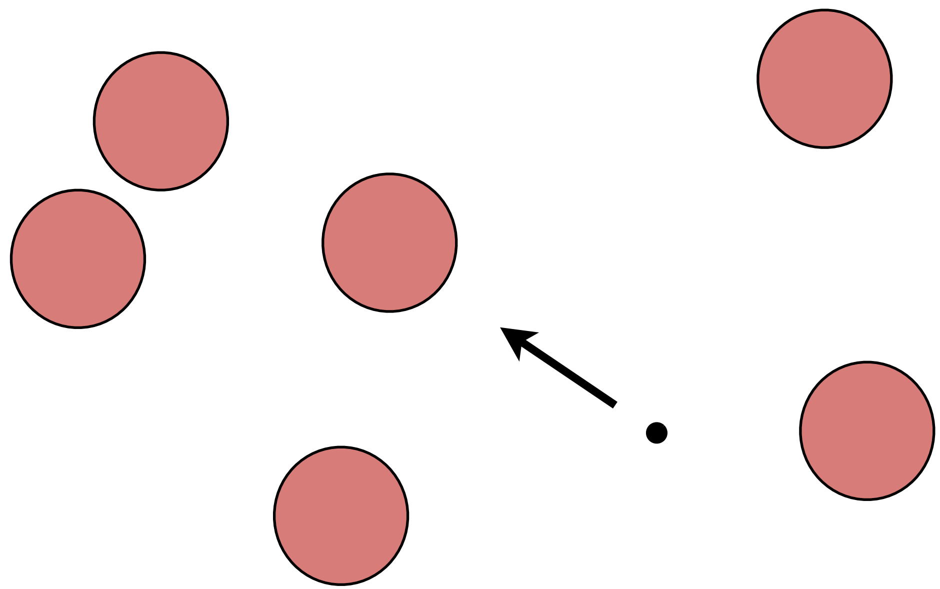
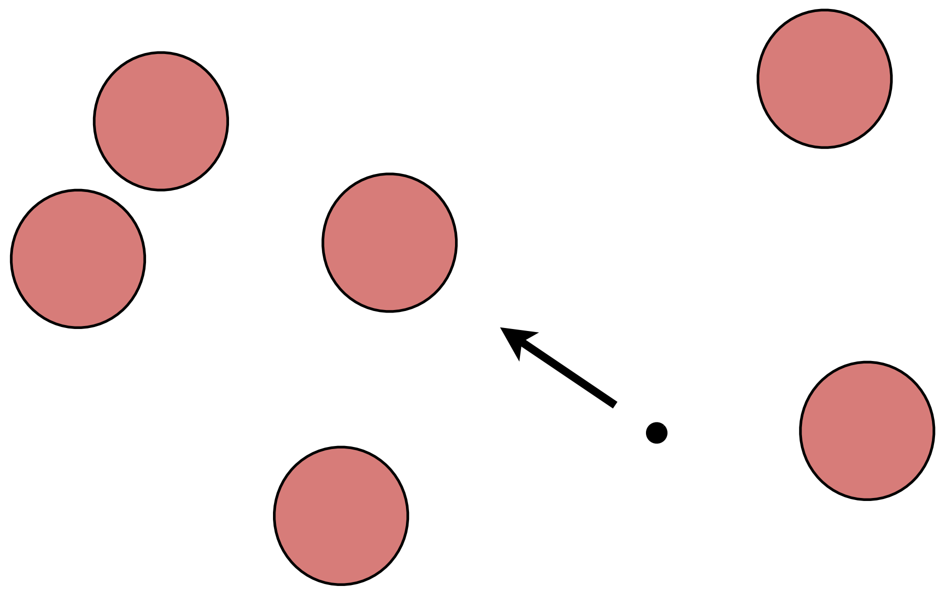


Boltzmann is based on successive 2-2 collisions



Boltzmann is based on successive 2-2 collisions

This microscopic picture is *also* what encodes chaotic trajectories

- A very special feature of dilute gases

Maxwell

$$\eta = \frac{1}{3} m \sqrt{\langle v^2 \rangle} \frac{1}{\sigma_{2-to-2}}$$

van Zon, van Beijeren, Dellago

$$\lambda = \frac{1}{\tau_{\rm ave}} \langle \frac{1}{2} \ln (\Delta \vec{v})^2 \rangle \simeq \frac{\sqrt{\langle v_{\rm rel}^2 \rangle}}{\ell_{\rm m.f.p.}} \simeq \rho \sqrt{\langle v^2 \rangle} \sigma_{\text{2-to-2}}$$

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- Transport follows from the Boltzmann equation

$$\frac{d}{dt} f(\mathbf{p}, t) = \int_{\mathbf{k}} (R^{in}(\mathbf{p}, \mathbf{k}) - R^{out}(\mathbf{p}, \mathbf{k})) f(\mathbf{k}, t)$$

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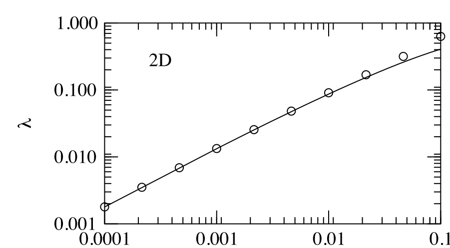
- Can we understand chaos from a kinetic-like equation?

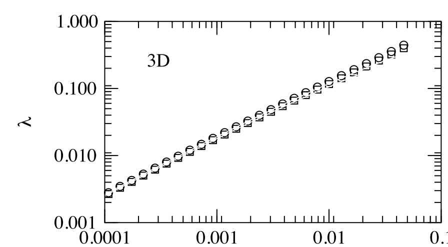
Ad hoc: clock equation

$$\frac{d}{dt} f_k = -f_k + f_{k-1}^2 + 2 f_{k-1} \sum_{\ell=0}^{k-2} f_\ell$$

van Zon, van Beijeren, Dorfman; Saarloos



f_k the fraction of particles which have experienced k collisions

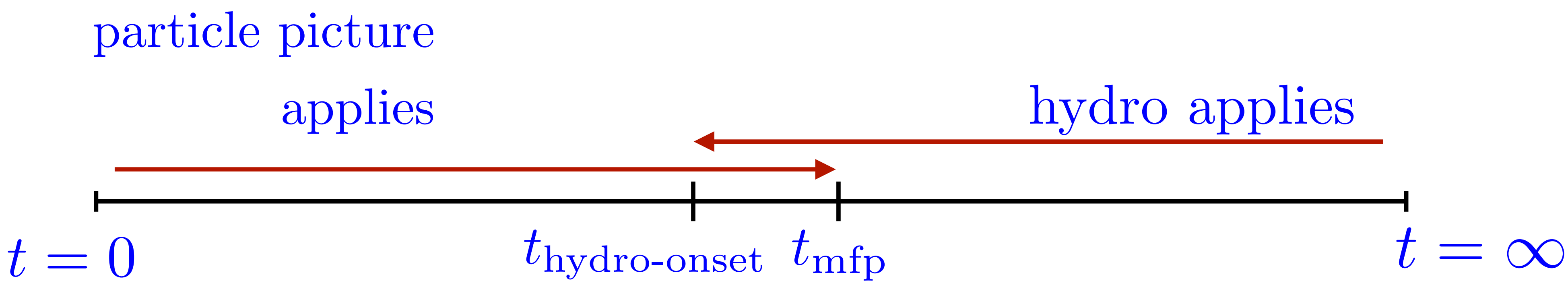


- Scrambling rate/Chaos is a microscopic “particle” property

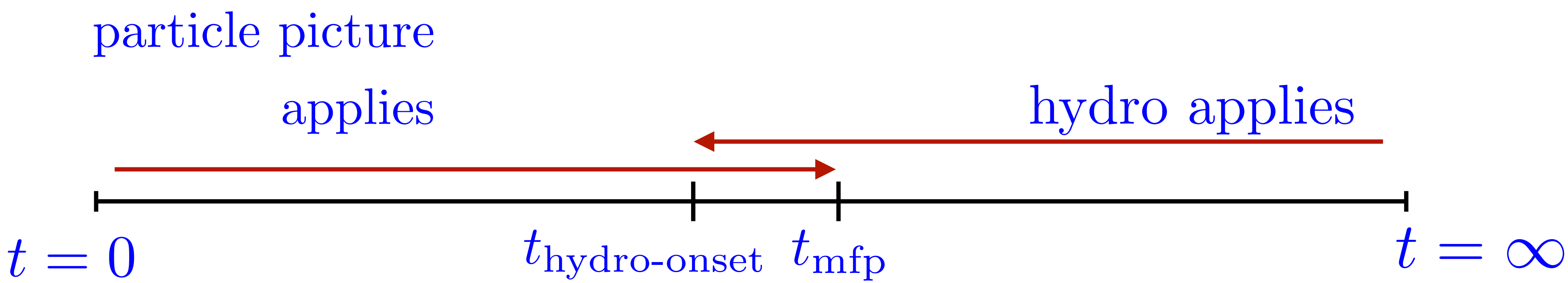
- Transport diffusion is a macroscopic collective property

- A generic system

- Special case: weakly coupled dilute gas



$$\eta = \frac{1}{3} m \sqrt{\langle v^2 \rangle} \frac{1}{\sigma_{2-to-2}}$$



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Implies hydro/Boltzmann/kinetic theory should also know about chaos!

$\text{scrambling=chaos=ergodicity}$ is very different from $\text{local therm.=equilibration}$

There is a connection:

In classical thermalization chaos is the source of ergodicity

In special situations (weakly coupled dilute gas) they are set by the same physics

~~Quantum~~ chaos from an out-of-time correlation function

Semi-classical

- A QFT way to detect chaos

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle$$

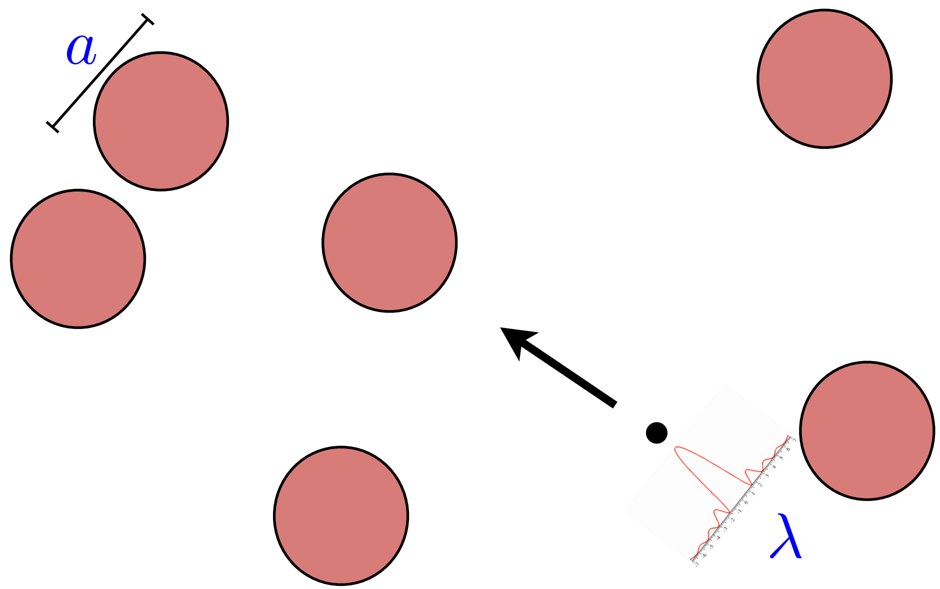
  - Choose

$$W = q(t) \quad V = p(0)$$

$$[W(t), V(0)] = [q(t), p(0)] = i\hbar\{q(t), p(0)\} = i\hbar \frac{\partial q(t)}{\partial q(0)}$$

$$\text{Chaos}: \; q(t) \sim \delta q(0) e^{\lambda_L t} \qquad C(t) \sim \hbar^2 e^{2\lambda t} \text{ with } \lambda = \lambda_{\text{Lya}}$$

- Semi-classical computation of conductivity in weak disorder



- Semiclassical regime $\lambda \ll a$

Larkin, Ovchinnikov

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \sim \hbar^2 e^{2\lambda t}$$

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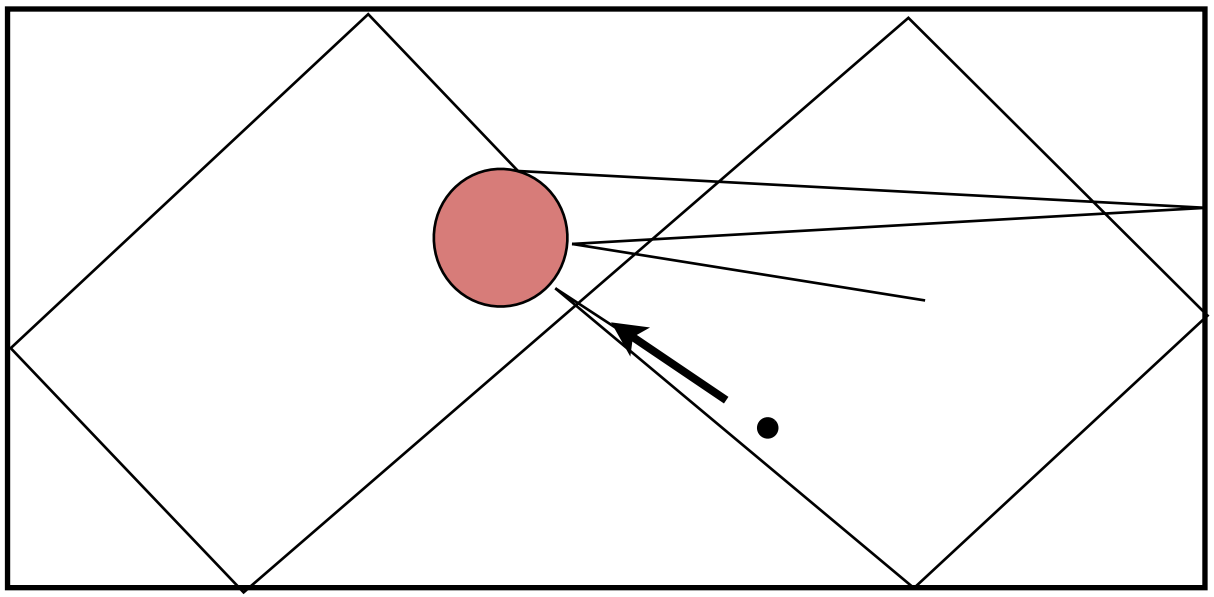

- Semiclassical regime $\lambda \ll a$ variation on Sinai billiards

Larkin, Ovchinnikov

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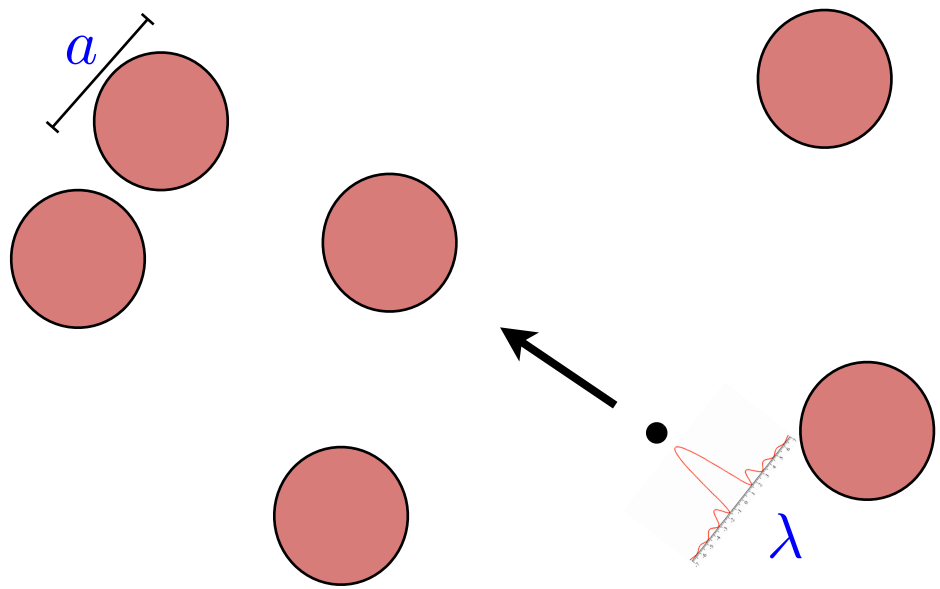

- Semiclassical regime $\lambda \ll a$
- Nevertheless: quantum physics takes over when

Larkin, Ovchinnikov

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \sim \hbar^2 e^{2\lambda t} \sim 1$$

Ehrenfest time: $t_{Ehr} = \frac{1}{\lambda} \ln \frac{1}{\hbar}$

- Careful:

  In the quantum regime chaotic behavior is hard.

  i.e. most quantum analogues of classical systems with chaos do not exhibit exponential growth in this OTOC correlator.

  - Need a small parameter — e.g. Grozdanov, Kukuljan, Prosen
  - In semi-classical systems $\hbar$ $\quad C(t) \sim \hbar^2 e^{2\lambda t}$
  - In holography: $\frac{1}{N}$ $\quad C(t) \sim \frac{1}{N^2} e^{2\lambda t}$

    Semi-classical single-trace lumps: large N classicalization/master field

A bound on chaos = a bound on diffusion?

- A bound on chaos

Maldacena, Shenker, Stanford

  - Related regulated function:

$$F(t) = \langle W(t) y V(0) y W(t) y V(0) y \rangle \sim 1 - e^{2\lambda t}$$

$$y^4 = \frac{e^{-\beta H}}{Z}$$

  - *Not time ordered:* but $|TFD\rangle = \sum_n e^{-\frac{\beta}{2}E} |n\rangle |n\rangle$

$$F(t) = \sum \langle TFD| (W(t)V(0) \otimes \mathbb{1})(1 \otimes W(t)V(0)) |TFD\rangle$$

$$F(t) \sim \sum \langle W(t)V(0)\rangle^\dagger \langle W(t)V(0)\rangle$$

  - Analyticity in QFT demands

$$\lambda \leq 2\pi T$$

- A bound on chaos

Maldacena, Shenker, Stanford

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Careful:
Answer depends on regulating.
This one encodes chaos correctly

Romero-Bermudez, Schalm, Scopelliti

  - Analyticity in QFT demands

$$\lambda \leq 2\pi T$$

- Black holes saturate this bound: maximal chaos

$$\lambda_{BH} = 2\pi T$$

  - This observation is the driving force behind SYK

Kitaev
e.g. Stanford@Strings'16

It would be nice to have a solvable model of holography.

theory	bulk dual	anom. dim.	chaos	solvable in $1/N$
SYM	Einstein grav.	large	maximal	no
$O(N)$	Vasiliev	$1/N$	$1/N$	yes
SYK	"$\ell_s \sim \ell_{AdS}$"	$O(1)$	maximal	yes

- A refined version

$$C(t,x) = -\langle [W(t,x), V(0)]^\dagger [W(t,x), V(0)] \rangle \sim \hbar^2 e^{\xi(x - v_{LR} t)}$$

gives you a "scrambling" velocity

$$\xi v_{LR} = 2\lambda$$

- First pioneered in 1+1 dimension systems
- Lieb-Robinson proved:

  The velocity v_{LR} is an absolute upper bound on information spreading.
- v_{LR} acts as en emergent lightcone.

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  - Lieb-Robinson proved:

    The velocity v_{LR} is an absolute upper bound on information spreading.
  - v_{LR} acts as en emergent lightcone.
- Idea: also in other systems this butterfly/Lieb-Robinson velocity is the maximum "speed" at which information spreads

- Diffusion is characterized by a velocity

$$D \sim \frac{v^2}{T} \sim \frac{v^2}{\lambda}$$

- Long sought goal: a fundamental quantum bound on diffusion

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

Kovtun, Son, Starinets

$$D \geq \frac{v_{inc}^2}{T}$$

Hartnoll
Hartman, Hartnoll, Mahajan

- (Unstated) Hypothesis: v_{LR} provides this fundamental velocity

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Kovtun, Son, Starinets

$$D \geq \frac{v_{inc}^2}{T} \quad \text{or} \quad D \leq \frac{v_{inc}^2}{T}$$

Hartnoll
Hartman, Hartnoll, Mahajan
Lucas,
......

- (Unstated) Hypothesis: v_{LR} provides this fundamental velocity

- Semi-classical chaos in weakly coupled systems

“Surprisingly a relation of the form $D \sim v_{LR}^2 \tau$ shows up in a number of non-holographic contexts”

- Most of these are weakly coupled zero density field theory results.

  This should not be a surprise. This is the classical dilute gas computation.

- Scrambling rate/Chaos is a microscopic “particle” property

- Diffusion is a macroscopic collective property

A kinetic equation for semi-classical chaos

- Semi-classical chaos in weakly coupled systems

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From the point of view what you compute it is a *surprise*

Scrambling in weakly coupled QFT is classical dilute gas

- Object of interest for λ, v_{LR}

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \quad \sim e^{2\lambda(t - \frac{x}{v_{LR}})}$$

growing mode

- Object of interest for $D = \dfrac{\eta}{\chi}$

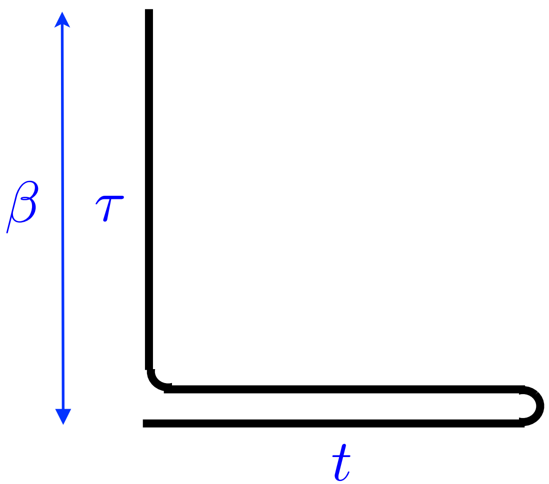
$$\eta = \lim_{\omega \to 0} \frac{1}{i\omega} \mathrm{Im} \langle T_{xy}(\omega), T_{xy}(-\omega) \rangle_R$$

Boltzmann transport only supports decaying modes:
viscosity set by smallest decay mode — relaxation time approximation

- Transport

$$G_R(t) \sim p_x p_y q_x q_y \langle [\Phi^{ab}\Phi^{ab}, \Phi^{cd}\Phi_{cd}] \rangle_\beta$$

Schwinger-Keldysh contour



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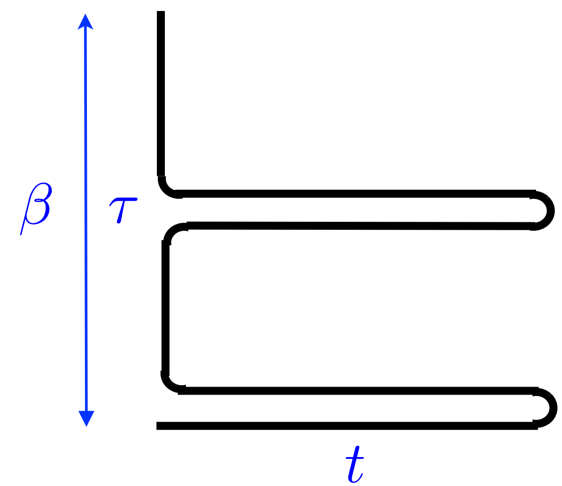
OTOC contour



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OTOC contour

- In free field theory

$$C(t) \sim G_R(t) = -2G_R^{\Phi\Phi}(t) + \mathcal{O}(\lambda)$$

Stanford, Jeon

- In perturbation theory Transport and Scrambling sum the same ladder diagrams

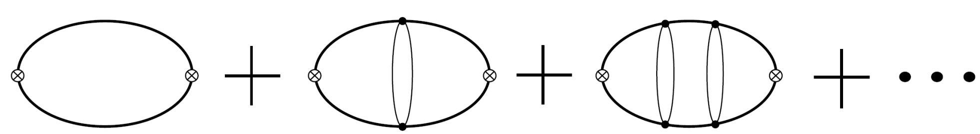

FIG. 2: Resummation of ladder diagrams. The insertions of the energy-momentum tensor operator $\hat{T}^{xy}$ is denoted by the crossed dots and black dots are the vertices with the coupling constant λ.

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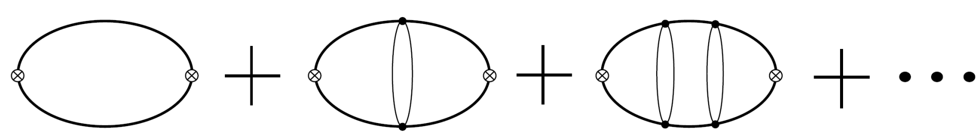

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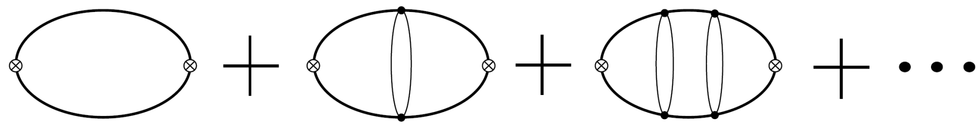
This Bethe-Salpeter eqn is the QFT version of the Boltzmann equation



$$\widetilde{G}(p|k) = \frac{\pi}{E_{\mathbf{p}}}\frac{\delta(p_0^2 - E_{\mathbf{p}}^2)}{-i\omega + 2\Gamma_{\mathbf{p}}}\left[1 + \int \frac{d^4\ell}{(2\pi)^4} R(\ell - p)\widetilde{G}(\ell|k)\right].$$

- Ansatz

$$\widetilde{G}(p|k) = \delta(p_0^2 - E_{\mathbf{p}}^2) f(\mathbf{p}|k)$$

$$(-i\omega + 2\Gamma_{\mathbf{p}}) f(\mathbf{p}|k) = \frac{\pi}{E_{\mathbf{p}}}\left[1 + \int_{\mathbf{l}} (R(E_{\mathbf{l}} - E_{\mathbf{p}}, \mathbf{l} - \mathbf{p}) + R(E_{\mathbf{l}} + E_{\mathbf{p}}, \mathbf{l} - \mathbf{p})) f(\mathbf{l}|k)\right].$$

gives

$$\frac{d}{dt} f(\mathbf{p}, t) = \int_{\mathbf{k}} (R^{in}(\mathbf{p}, \mathbf{k}) - R^{out}(\mathbf{p}, \mathbf{k})) f(\mathbf{k}, t)$$

Schwinger Keldysh vs OTOC Contour

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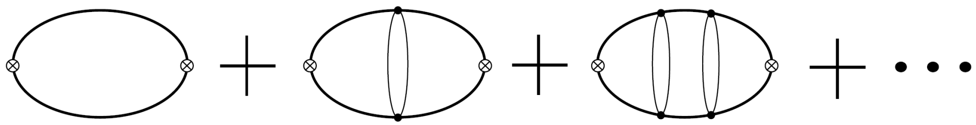
- SchwKeld 

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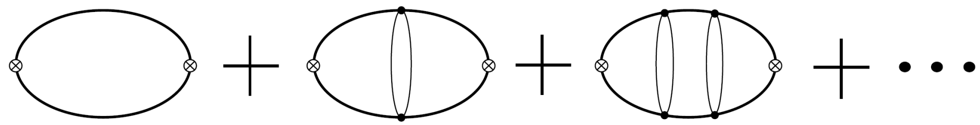
- SchwKeld 

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Grozdanov, Schalm, Scopelliti,

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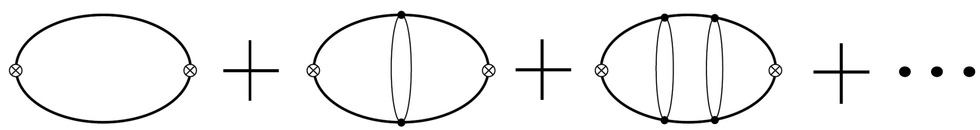


Boltzmann equation (net density)

$$\frac{d}{dt} f(\mathbf{p},t) = \int_{\mathbf{k}} (R^{in}(\mathbf{p},\mathbf{k}) - R^{out}(\mathbf{p},\mathbf{k})) f(\mathbf{k},t)$$

purely relaxational

$$f(\mathbf{p},t) \sim e^{\lambda t} \text{ with } \lambda \leq 0$$

Kinetic equation (gross collisions)*

$$\frac{d}{dt} \mathsf{f}(\mathbf{p},t) = \int_{\mathbf{k}} \frac{\epsilon(\mathbf{p})}{\epsilon(\mathbf{k})} (R^{in}(\mathbf{p},\mathbf{k}) + \widehat{R^{out}}(\mathbf{p},\mathbf{k})) \mathsf{f}(\mathbf{k})$$

front propagation into unstable states

$$\mathsf{f}(\mathbf{p},t) \sim e^{\lambda t} \text{ with } \lambda \leq \lambda_{max} > 0$$

Saarloos, vBeijeren,
Aleiner, Faoro, Ioffe

$$* : \widehat{R^{out}}(\mathbf{p},\mathbf{k}) = R^{out}(\mathbf{p},\mathbf{k}) - 2\delta(\mathbf{p}-\mathbf{k}) R^{out}(\mathbf{k},\mathbf{k})$$

- Chaos follows from kinetic equation for *gross* energy exchange

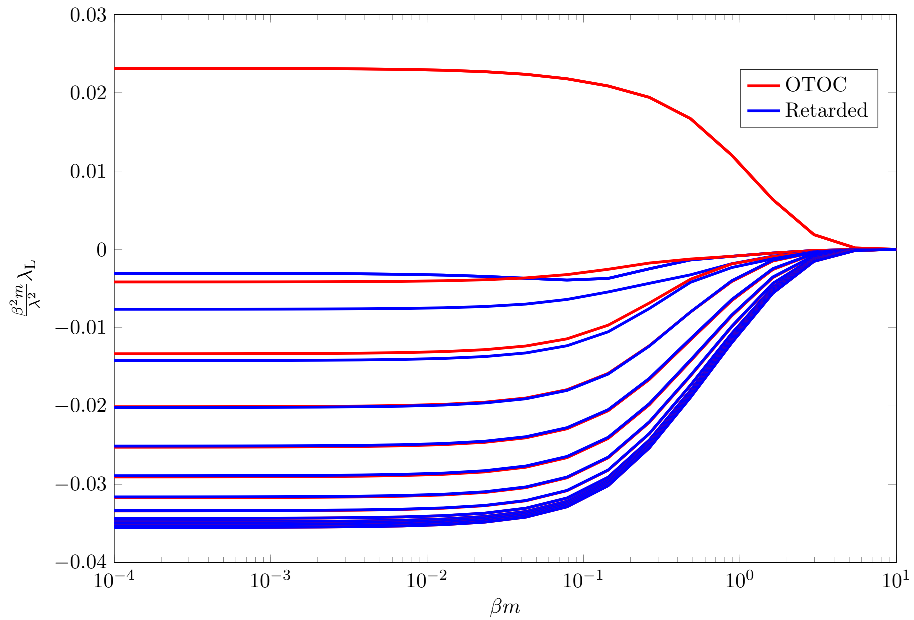
$$\frac{d}{dt}\mathsf{f}(\mathbf{p},t) = \int_{\mathbf{k}} \frac{\epsilon(\mathbf{p})}{\epsilon(\mathbf{k})} \left(R^{in}(\mathbf{p},\mathbf{k}) + R^{out}(\mathbf{p},\mathbf{k}) - 2\delta(\mathbf{p}-\mathbf{k})R^{out}(\mathbf{k},\mathbf{k})\right)\mathsf{f}(\mathbf{k})$$

  - This is derived as opposed to ad hoc clock model

$$\frac{d}{dt}f_k = -f_k + f_{k-1}^2 + 2f_{k-1}\sum_{\ell=0}^{k-2} f_\ell$$

Qualitatively physics is similar (unstable front dynamics) blue: eigenvalues λ for SchwKeld/Boltzmann

red: eigenvalues λ for OTOC/Energy-exchange



- This explicitly shows in weakly coupled dilute QFT scrambling and diffusion are set by the same dynamics --- even though they are not identical.

$$\text{blue: eigenvalues } \lambda \text{ for SchwKeld/Boltzmann}$$

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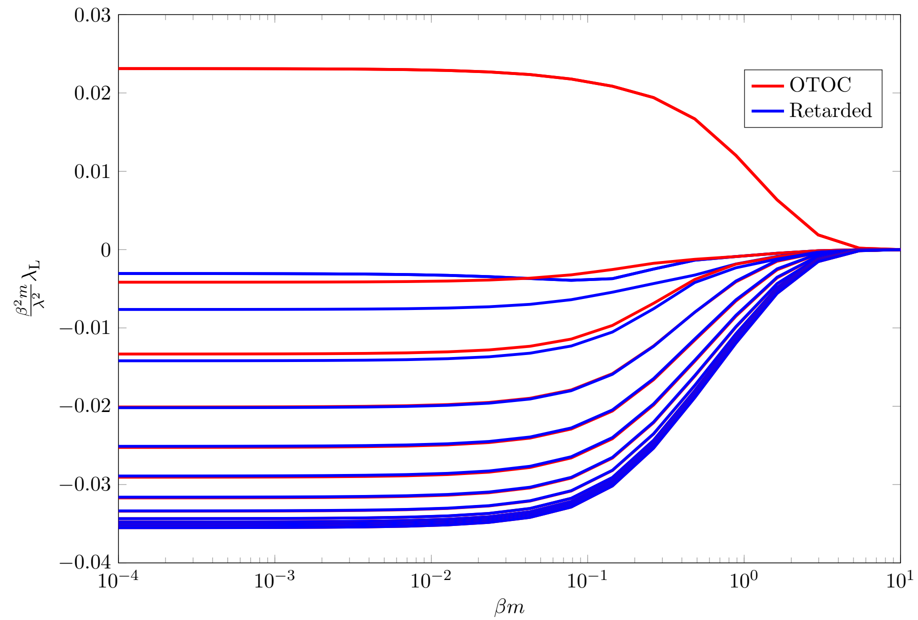

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$$\eta = \frac{1}{3} m \sqrt{\langle v^2 \rangle} \frac{1}{\sigma_{2-to-2}}$$

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- We have now shown that this holds in general:
  - For bosonic and fermionic systems (Gross-Neveu model)
  - Models near a QCP approached from perturbative regime (Wilson-Fisher $O(N)$ model)
  - Shorter derivation using 2PI formalism
- In all cases *off-shell* Bethe-Salpeter contains both chaos and Boltzmann transport.
  - One solution ansatz: Boltzmann. Complement: Chaos
  - pQFT analogue of Maxwell relation: weakly coupled dilute gas.
  - Pole-skipping….

Grozdanov, Schalm, Scopelliti, arXiv:1912.xxxx

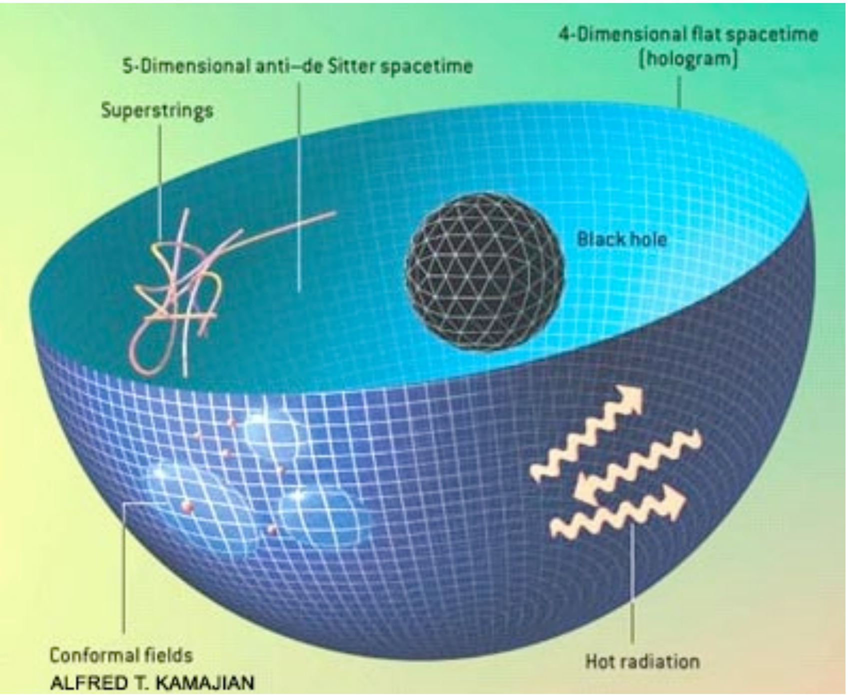
Ultra strongly correlated systems are similar to dilute gases

- Is scrambling rate related to diffusion?

$$D \sim \frac{v^2}{T} \sim \frac{v_{\mathrm{LR}}^2}{\lambda}$$

AdS-CFT duality

strongly coupled field theories without an energy scale (CFT) have a dual description as a weakly coupled string theory in negatively curved space time (AdS).

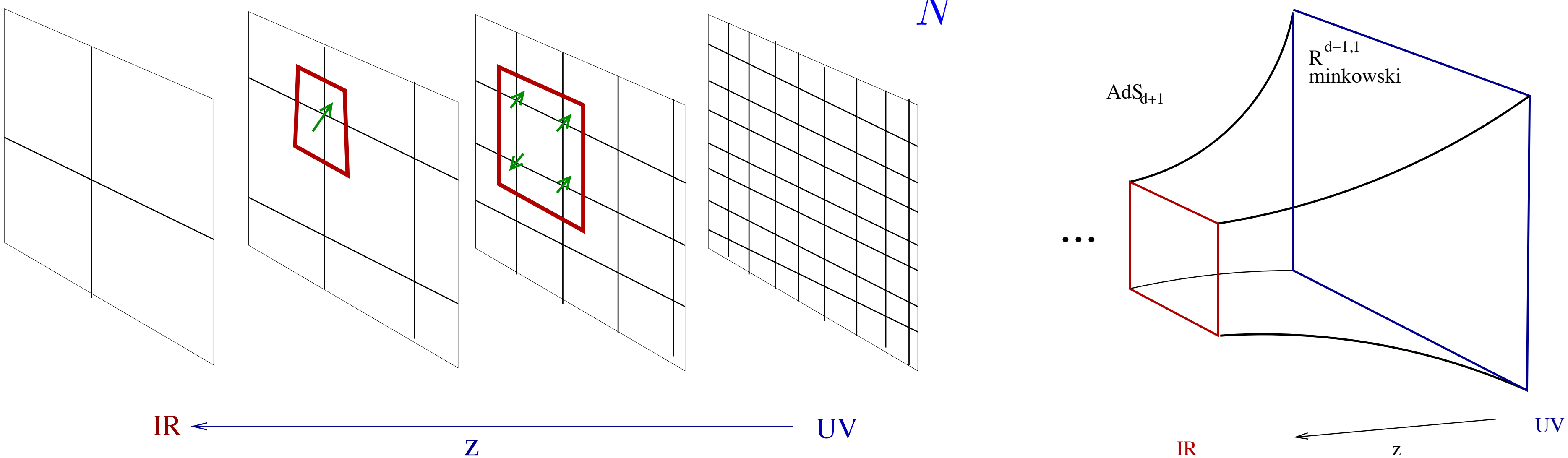


Maldacena ATMP**2**, 231 (1998); Witten ATMP**2**, 253 (1998); Gubser, Klebanov, Polyakov, PL**B428**,105 (1998)

Holography for Strongly coupled systems

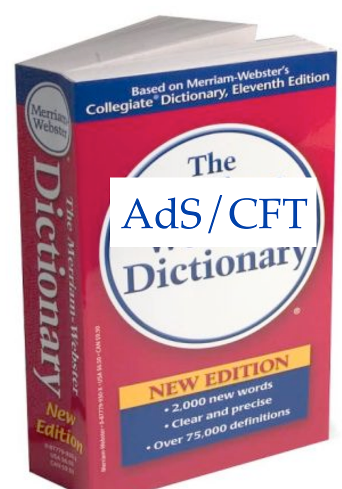
works best when d.o.f. are matrices Φ_{ij} $i,j = 1 \dots N$ with $N \gg 1$

semi-classical limit $\frac{1}{N} \to 0$



$$Z_{CFT}(J) = \exp i S_{AdS}^{on-shell}(\phi(\phi_{\partial AdS} = J))$$

Quantum numbers
Finite Temp
Finite Density
Conserved Current
Energy dynamics



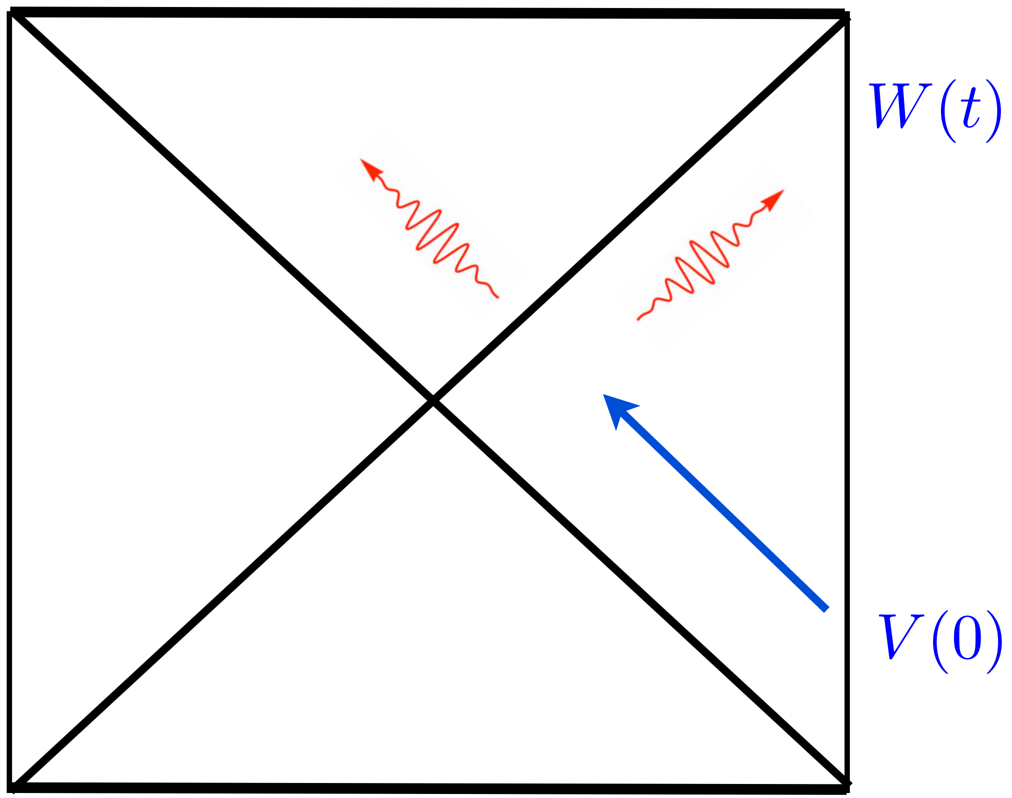
Quantum numbers
AdS Black hole
Extremal AdS black hole
Gauge field
Gravity dynamics

OTOC in holography

- Shockwave calculation in AdS BH

Roberts, Stanford, Susskind

$$F(t) = \sum \langle TFD|(W(t)V(0) \otimes \mathbb{1})(1 \otimes W(t)V(0))|TFD\rangle$$



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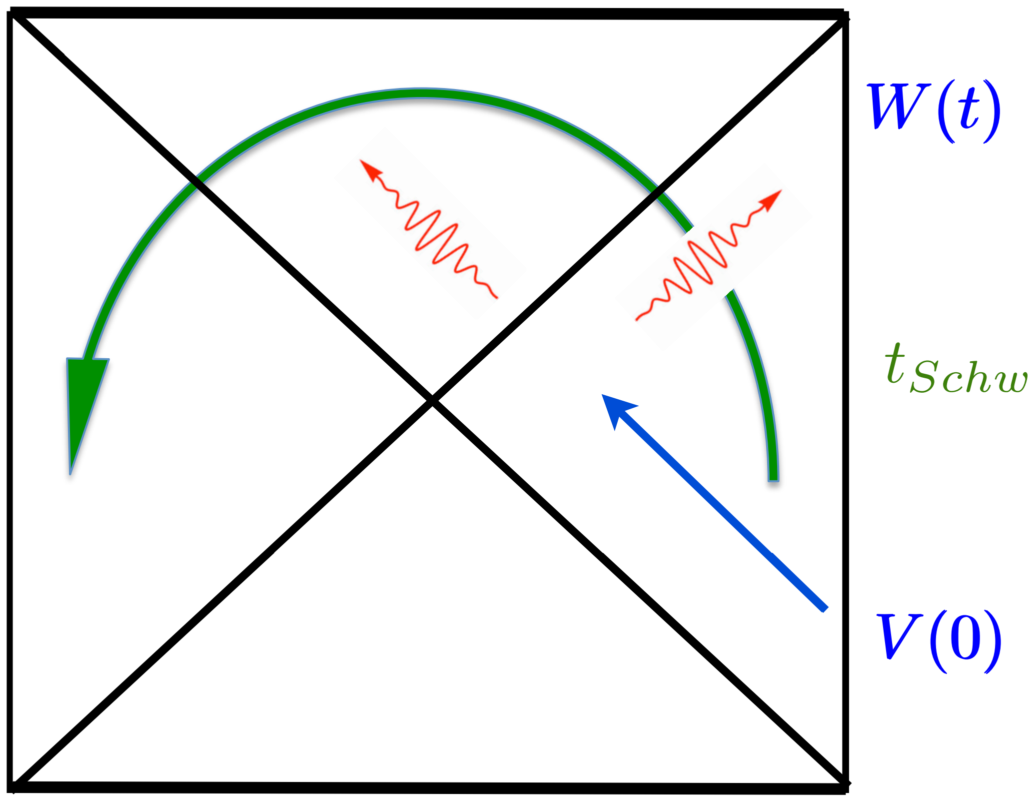


- Is scrambling rate related to diffusion?

$$D \sim \frac{v^2}{T} \sim \frac{v_{\mathrm{LR}}^2}{\lambda}$$

Blake;
Davison, Fu, Georges, Gu, Jensen, Sachdev.

- Is scrambling rate related to diffusion?

For "relevant diffusion" (=irrelevant suscep)

$$D = \frac{d-\theta}{\Delta_\chi} \frac{v_{LR}^2}{2\pi T}$$

$$\Delta_\chi \equiv [\rho] - [\mu] > 0$$

..similar results for massive gravity (mean-field disorder), but fails in general

Lucas, Steinberg;
Gu, Lucas, Qi

- Refinement: charged systems with mean-field disorder
  - Thermal diffusivity set by horizon properties only

$$D_P = \eta / sT$$

Policastro, Son, Starinets

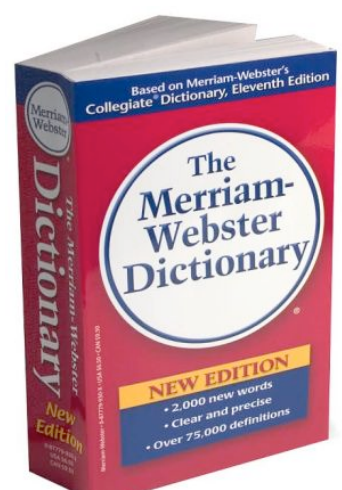
$$D_T = \frac{z}{2z-2} \frac{v_{LR}^2}{\lambda_L}$$

Blake, Davison, Sachdev

- From a physics perspective these are puzzling results:

$$Z_{CFT}(J) = \exp i S_{AdS}^{on-shell}(\phi(\phi_{\partial AdS} = J))$$

Quantum numbers
Finite Temp
Finite Density
Conserved Current
Energy dynamics



Quantum numbers
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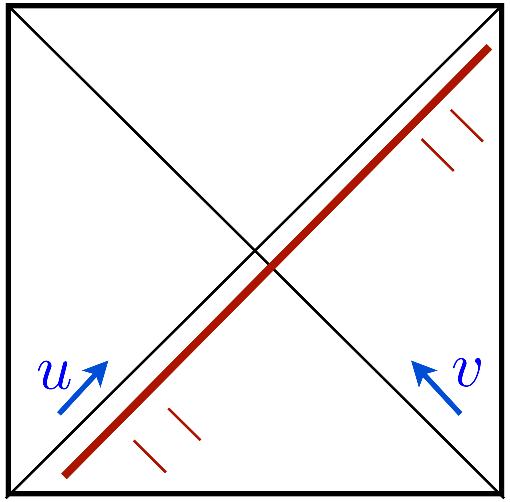
- Shock waves are sound
  - General metric

$$ds^2_{d+2} = A(UV)dU dV + B(UV)g_{ij}dx^i dx^j - A(U,V)h(U,\vec{x})dU dU$$

  - Shock wave equation

$$\delta(U)\left(\Delta_g h - d\frac{B'}{A}h\right) = 32\pi E A \delta^d(\vec{x})\delta(U)$$



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  - Sound perturbation from AdS/CFT

$$\Delta_g h(U,\vec{x}) - 2d\frac{B}{A}h(U,\vec{x}) - d\frac{B'}{A}U\frac{\partial}{\partial U}h(U,\vec{x}) = 0$$

$$\text{for } h(U,\vec{x}) \sim \delta(U)h(\vec{x}) \text{ reduces to shock}$$

- The shockwave is in Kruskal coordinates.
  - Using Poincare coordinates

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2 - e^{ikz}\left(f(r)H_1(t,r)dt^2 - 2H_2(t,r)dtdr + H_3(t,r)\frac{dr^2}{f(r)}\right).$$

  - Solution to Einstein's Eqns:

$$H_1(t,r) = H_3(t,r) = \left(C_1\, e^{\frac{k^2 t}{3r_+}} + C_2\, e^{-\frac{k^2 t}{3r_+}}\right) e^{-\frac{k^2+12r_+^2}{3r_+}\int^r dr' f(r')^{-1}},$$

$$H_2(t,r) = \left(C_1\, e^{\frac{k^2 t}{3r_+}} - C_2\, e^{-\frac{k^2 t}{3r_+}}\right) e^{-\frac{k^2+12r_+^2}{3r_+}\int^r dr' f(r')^{-1}}.$$

- Write as a sound wave.
  - Obeys a diffusion relation

$$\omega_o = \frac{ik^2}{3r_+}, \ \omega_i = -\frac{ik^2}{3r_+},$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2 - C_1\, e^{-i\omega_o t + ikz} e^{(i\omega_o - 4r_+)r_*(r)} f(r) \left(dt - \frac{dr}{f(r)}\right)^2$$

$$- C_2\, e^{-i\omega_i t + ikz} e^{-(i\omega_i + 4r_+)r_*(r)} f(r) \left(dt + \frac{dr}{f(r)}\right)^2 .$$

- For the sound wave to be regular (on the horizon)

$$\omega_o = -2ir_+ = -2i\pi T\,, \quad \omega_i = 2ir_+ = 2i\pi T\,,$$

$$ds^2 = -\,f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2 - C_1\, e^{-i\omega_o(t+r_*(r))+ikz} f(r) \left(dt - \frac{dr}{f(r)}\right)^2 - C_2\, e^{-i\omega_i(t-r_*(r))+ikz} f(r) \left(dt + \frac{dr}{f(r)}\right)^2 .$$

- This regularity condition also means

$$k^2 + \mu^2 = 0, \;\; \text{with}\; \mu^2 = 6r_+^2 = 6\pi^2 T^2\,,$$

- This is the shock wave equation

$$\left(\partial_i \partial_i - \mu^2\right) h(x) = 0$$

- More precisely:
  - Sound is the physical (gauge-invariant) mode of h_{tt}
  - In radial gauge

$$Z_3 = h_{tt} + \left(\frac{k^2 f' - 2\omega^2 r}{2k^2 r}\right)(h_{xx} + h_{yy}) + \frac{2\omega}{k} h_{tz} + \frac{\omega^2}{k^2} h_{zz}$$

  - In a different gauge

$$Z_3 = h_{tt} - \frac{2i\omega f}{f'} h_{tr} + \frac{f^2}{f'^2}\left(2\omega^2 + f'^2\right) h_{rr}.$$

  - The latter reduces on the horizon to the previous calculation

Support is $1/U \text{ instead of } \delta(U)$

- Sound at *imaginary* values of frequency and momentum

$$\omega = 2\pi i T = i\lambda \quad , \quad k^2 = -\mu^2 = -6\pi^2 T^2 = -\frac{\lambda^2}{v_B^2}$$

- Hydrodynamical sound (known up to 3rd order analytically)

$$\omega(k) = \pm\frac{1}{\sqrt{3}}k - \frac{i}{6\pi T}k^2 + \ldots$$

  - Relaxational modes: real momentum, complex/imaginary frequency

    measures relaxation time

  - Penetration depth: real frequency, complex/imaginary momentum

    measures relaxation length (penetration depth)

  - Doubly imaginary: "temporal response" to "spatial profile"

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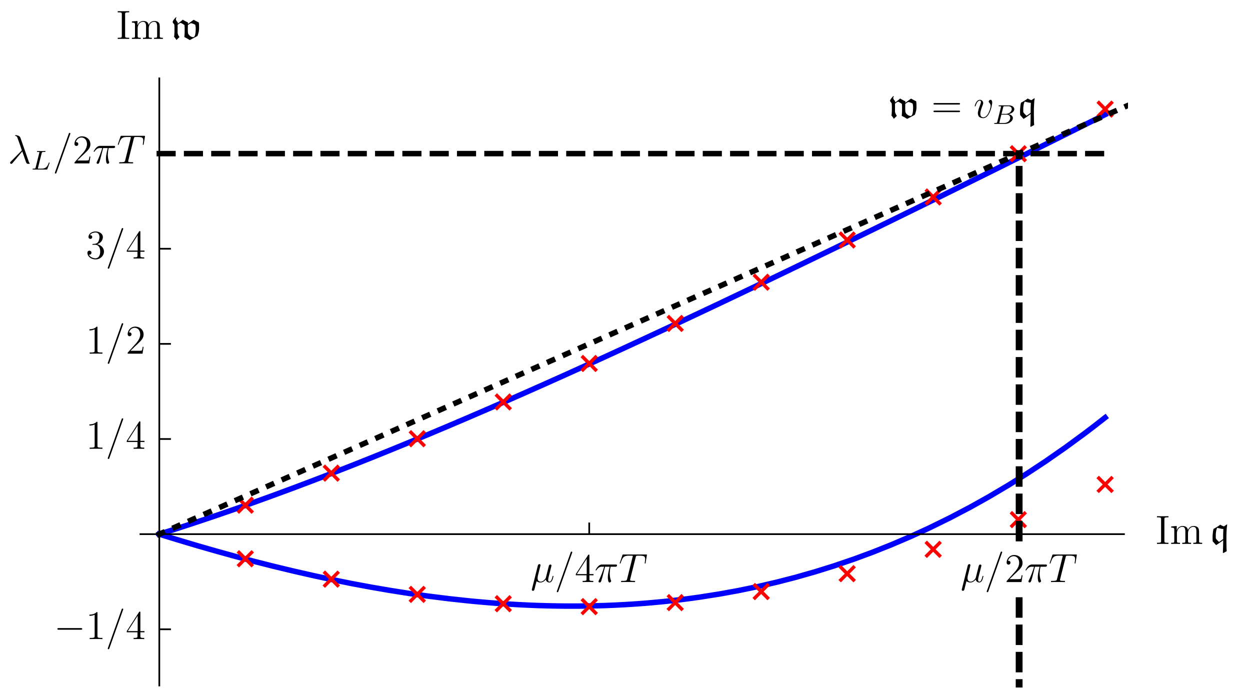

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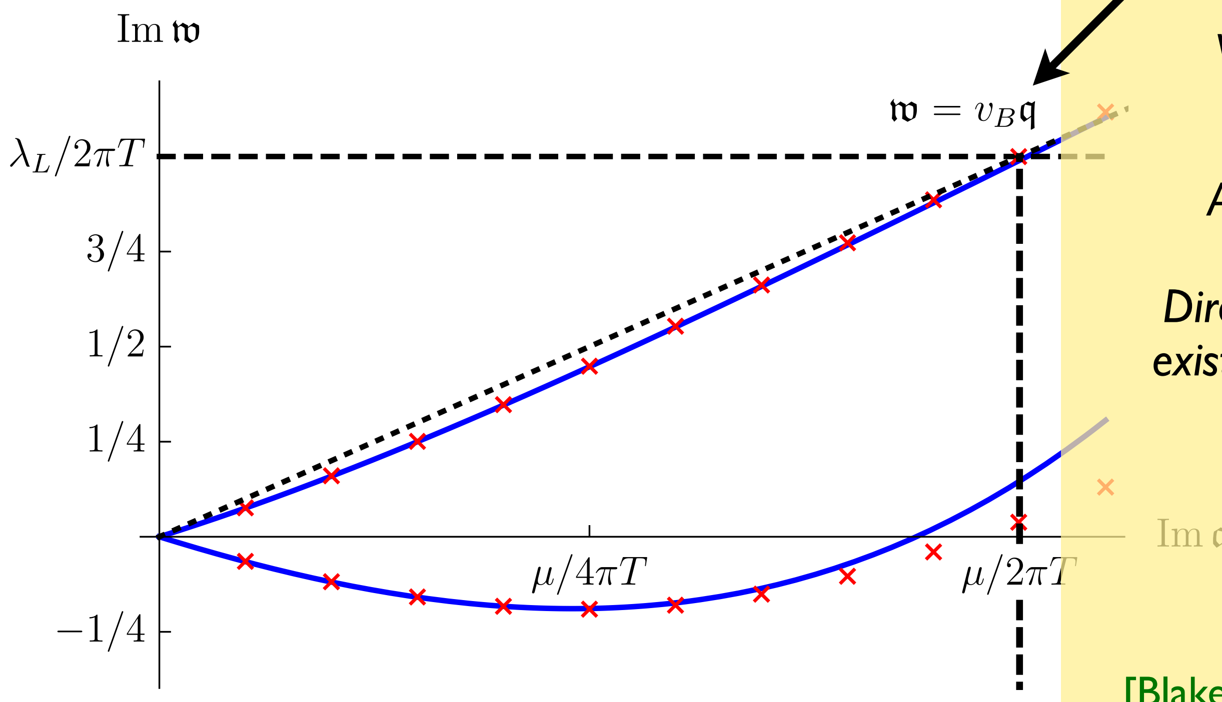

Pole-skipping:

QNM mode residue vanishes precisely at

$$\omega = 2\pi i T$$

Also happens in SYK.

[Gu, Qi, Stanford]

Direct consequence of the existence of the shockwave solution.

[Blake, Lee, Liu]

Beautiful GR story: non-unique BC at the horizon

[Blake, Davison, Grozdanov, Liu]

- In generality

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left[R + \frac{12}{L^2} + \mathcal{L}_{matter}\right]$$

$$ds^2 = -f(r)dt^2 + \frac{g(r)dr^2}{f(r)} + b(r)\left(dx^2 + dy^2 + dz^2\right) - \left[f(r)C_\pm W_\pm (dt \pm \frac{1}{f(r)}dr)^2\right]$$

$$W_\pm(t,z,r) = e^{-i\omega\left[t\pm\int^r \frac{dr'}{f(r')}\right]+ikz} h_\pm(r)$$

$$\partial_t W \pm |_{r_h} = \mp\mathfrak{D}\,\partial_z^2 W_I|_{rh} \quad tr\text{-Einstein Eq.}$$

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  - This does not equal the diffusion constant in the CFT

$$D_{CFT} = \frac{\eta}{sT} = \frac{3}{4} D_{hor} \qquad \frac{D}{\mathfrak{D}} = \frac{3\, b'(r_h)}{8\pi T} \,,$$

  - Even though this also computed on the horizon (special to momentum diffusion)

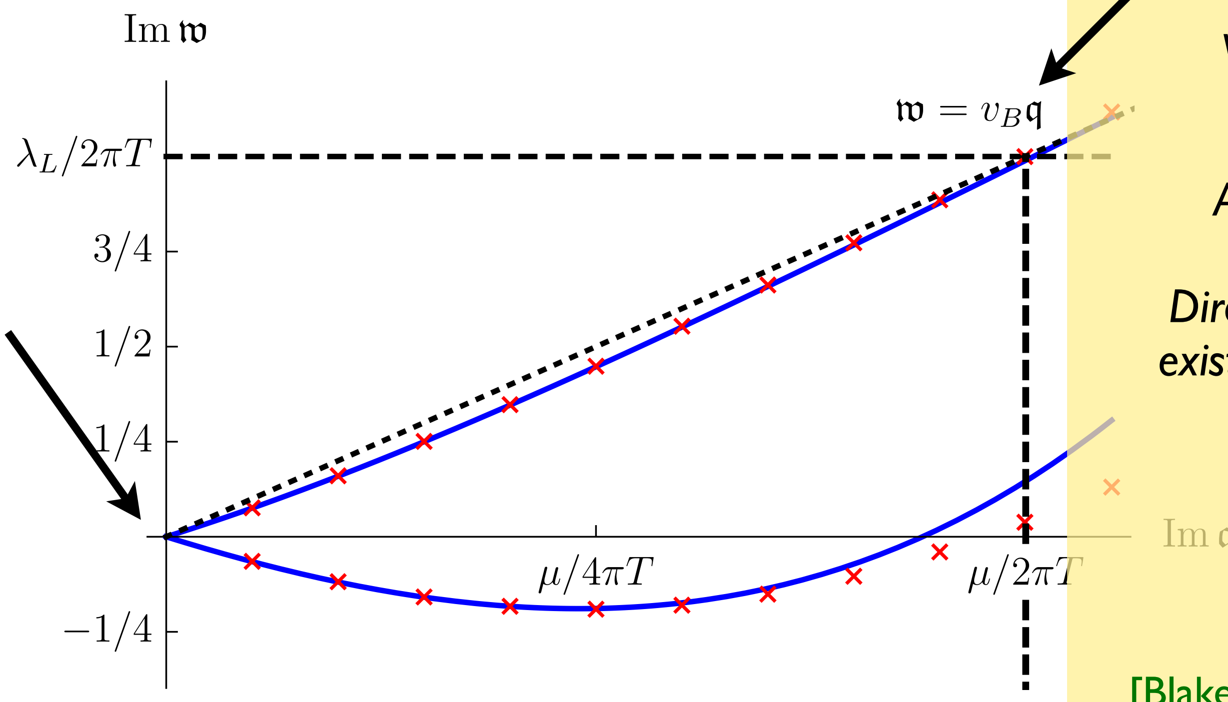
Davison, Fu, Georges, Gu, Jensen, Sachdev.
Blake, Davison, Sachdev

Physical diffusion is given by the behavior near

$\omega \ll 1$

by now verified in many models

[Blake, Davison, Grozdanov,Liu]



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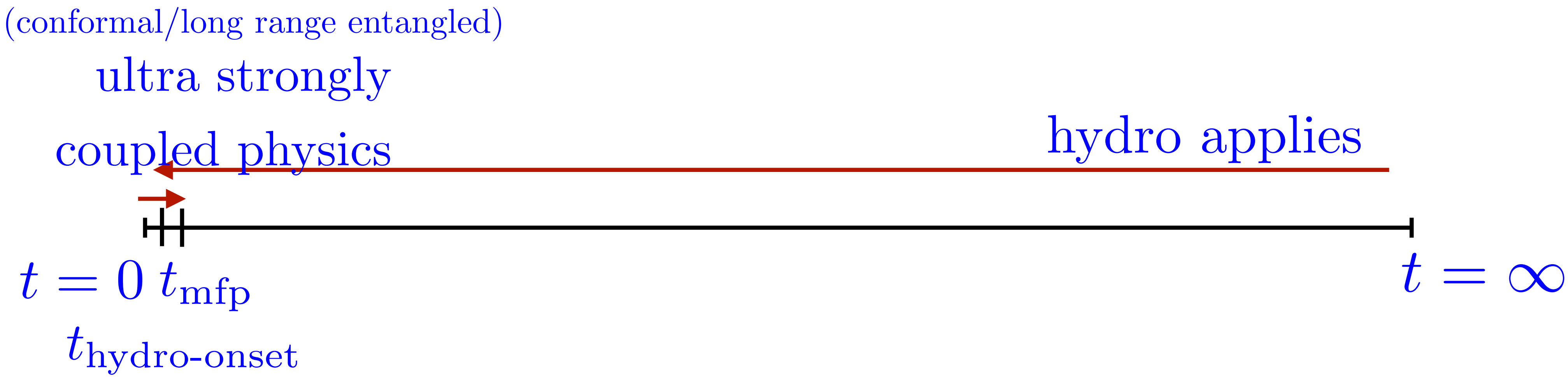
- A generic system


(conformal/long range entangled)
ultra strongly
coupled physics
hydro applies
$t = 0$
t_{mfp}
$t_{\text{hydro-onset}}$
$t = \infty$

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Davison, Fu, Georges, Gu, Jensen, Sachdev. Blake, Davison, Sachdev; Blake, Davison, Grozdanov, Liu

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Maxwell

$$\eta = \frac{1}{3} m \rho \ell_{\mathrm{m.f.p.}} \sqrt{\langle v^2 \rangle}$$

Famous "first" result of molecular kinetic theory

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van Zon, van Beijeren, Dellago

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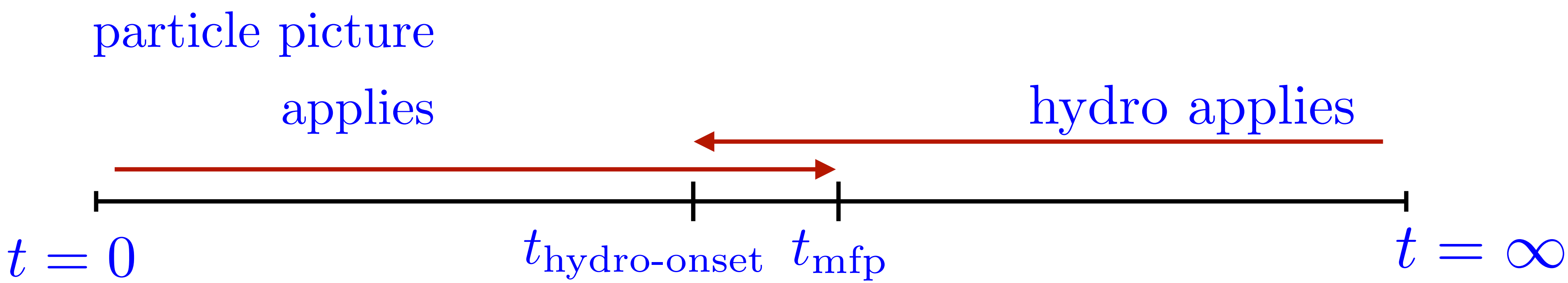
- Except: two-derivative holography

but now it is the macroscopic properties that set ergodicity

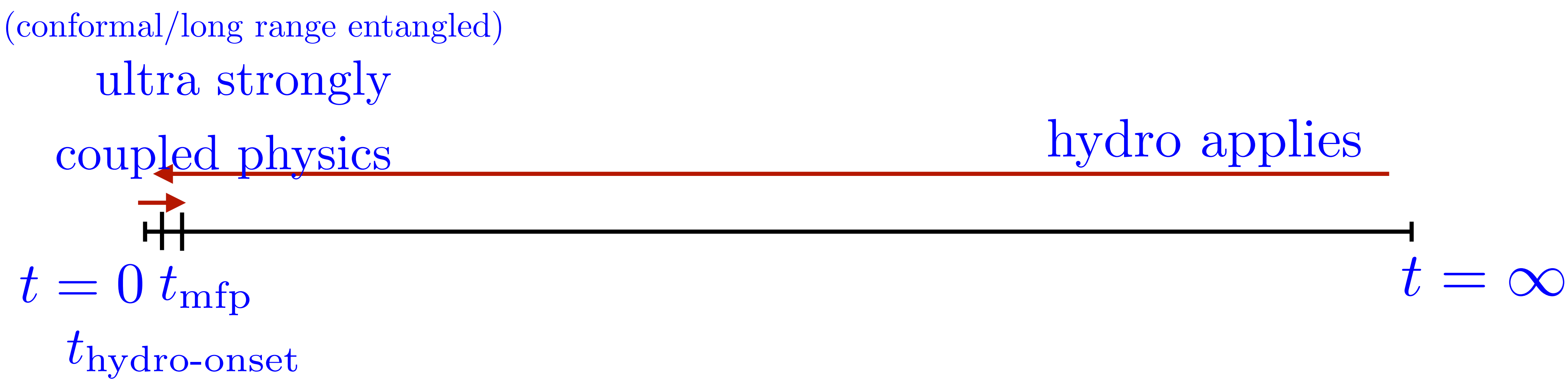

particle picture
applies
hydro applies
$t = 0$
t_{mfp}
$t_{\text{hydro-onset}}$
$t = \infty$



$$\eta = \frac{1}{3} m \sqrt{\langle v^2 \rangle} \frac{1}{\sigma_{2-to-2}}$$

And there is also a kinetic equation computing chaos!

$$\frac{d}{dt}\mathsf{f}(\mathbf{p}, t) = \int_{\mathbf{k}} \frac{\epsilon(\mathbf{p})}{\epsilon(\mathbf{k})} \left(R^{in}(\mathbf{p}, \mathbf{k}) + R^{out}(\mathbf{p}, \mathbf{k}) - 2\delta(\mathbf{p} - \mathbf{k}) R^{out}(\mathbf{k}, \mathbf{k}) \right) \mathsf{f}(\mathbf{k})$$



Ultra strongly coupled systems are similar to weakly coupled dilute gases: chaos and transport are set by the same physics.

Conclusion

1. Quantum Chaos from an out-of-time-correlation function

$$C(t) = -\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \sim \hbar^2 e^{2\lambda t} \sim 1$$

2. Chaos and diffusion

different time scales: exception dilute gas

3. A bound on chaos = a bound on diffusion?

No, here, or trivial, or ...

4. Ultra strongly correlated systems are similar dilute gases

Scrambling and diffusion are set by the same **semi-classical** physics.

5. A kinetic equation for semi-classical chaos

Grozdanov, Schalm, Scopelliti,
in graphene: Klug, Scheurer, Schmalian

$$\frac{d}{dt}\mathsf{f}(\mathbf{p},t) = \int_{\mathbf{k}} \frac{\epsilon(\mathbf{p})}{\epsilon(\mathbf{k})} \big(R^{in}(\mathbf{p},\mathbf{k}) + R^{out}(\mathbf{p},\mathbf{k}) - 2\delta(\mathbf{p}-\mathbf{k}) R^{out}(\mathbf{k},\mathbf{k}) \big)\, \mathsf{f}(\mathbf{k})$$

Thank you